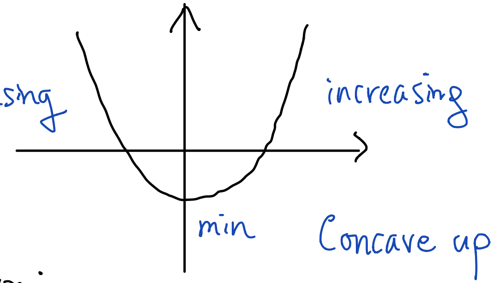
Math 2010 Advanced Calculus I

Differential Calculus of Multi-variable Functions

For one-variable function:

We study $f(x)$, e.g. $f(x) = x^2 - 1$

Differentiation: $f'(x) = 2x$, $f''(x) = 2$



Application:

- Curve Sketching
- Rate of Change: velocity, acceleration...
- Finding max / min
- Approximation

How about vector-valued multi-variable functions?

Objects of studies: $\vec{f}(\vec{x}) = \vec{f}(x_1, x_2, \cdots, x_n)$

e.g.

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\qquad g: \overbrace{\mathbb{R}^2}^{\text{multi-variable}} \longrightarrow \overbrace{\mathbb{R}^2}^{\text{vector-valued}}$

$f(x,y,z) = x \sin y \cos z$ $\qquad g(x,y) = (x^2 + y, e^{xy})$

Want to do similar things:

- Study graph of functions
- Rate of Change with respect to each variable or any direction
- Finding max / min
- Approximation

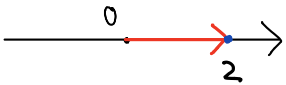
Euclidean Space $\mathbb{R}^n$ (Au 1.1, Thomas 12.1-12.3)

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n copies of $\mathbb{R}$)

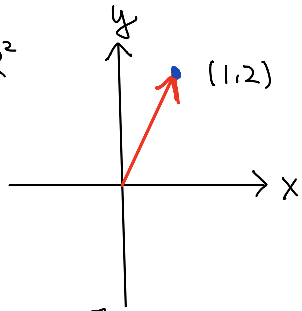
$$= \{\underbrace{(x_1, x_2, \cdots, x_n)}_{\text{Cartesian / Rectangular coordinates}} : x_i \in \mathbb{R} \text{ for } 1 \le i \le n\}$$

Pictures

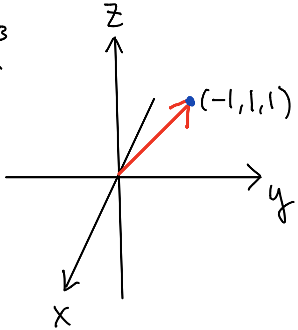
$\mathbb{R}^1$



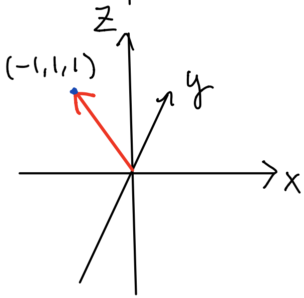
$\mathbb{R}^2$



$\mathbb{R}^3$



or



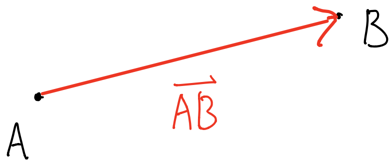
$\mathbb{R}^4$ or higher dimension: Difficult to draw

Rmk

Each $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ can be viewed as a point or a vector in $\mathbb{R}^n$.

Notations:

1. $\vec{x} = \mathbf{x} = x = (x_1, x_2, \cdots, x_n)$ (bold)
2. $\overrightarrow{AB}$ is a vector with
   initial point A, terminal point B
   length / magnitude $|\overrightarrow{AB}|$ or $\|\overrightarrow{AB}\|$



Basic operations of vectors

Let $\vec{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$

$\vec{b} = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$

$r \in \mathbb{R}$

Equality

$$\vec{a} = \vec{b} \iff a_i = b_i \text{ for } i = 1, 2, \cdots, n$$

Addition

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n)$$

Scalar multiplication

$$r\vec{a} = (ra_1, ra_2, \cdots, ra_n)$$

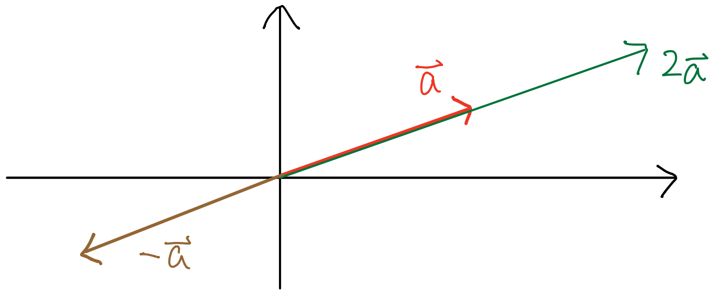
Subtraction

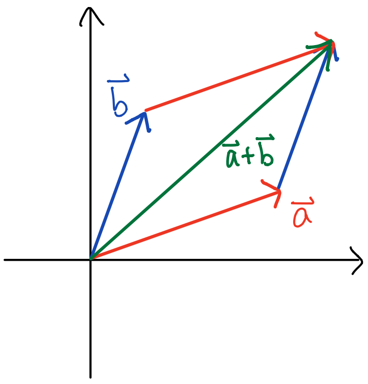
$$\vec{a} - \vec{b} = a + (-1)\vec{b}$$

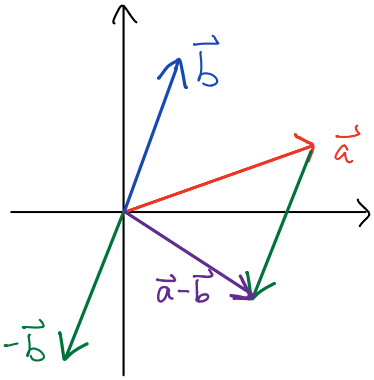
$$= (a_1 - b_1, a_2 - b_2, \cdots, a_n - b_n)$$

Pictures

$\vec{a} = (3, 1)$ $\vec{b} = (1, 2)$







Parallelogram law

Length, Dot Product in $\mathbb{R}^n$

Defn

Let $\vec{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$
$\quad\;\; \vec{b} = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$

Define **dot product**

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Warning: Don't forget the dot!

In particular,

$$\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + \cdots + a_n^2$$
$$= \|\vec{a}\|^2$$

Another notation

$$\underbrace{\langle \vec{a}, \vec{b} \rangle}_{\text{(Standard) inner product}} = \vec{a} \cdot \vec{b}$$

Properties of Dot Product

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$, $r \in \mathbb{R}$. Then

① $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

$\quad \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

② $(r\vec{a}) \cdot \vec{b} = a \cdot (r\vec{b}) = r(\vec{a} \cdot \vec{b})$

③ $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

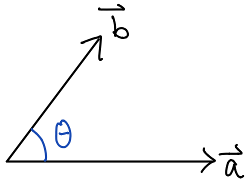
④ $\vec{a} \cdot \vec{a} \geq 0$

zero vector

with equality holds $\Leftrightarrow \vec{a} = \vec{0} = (0, 0, 0, \cdots, 0)$

⑤ $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

⑥ $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$



Hence, if $\vec{a}, \vec{b} \neq \vec{0}$,

$$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \vec{a} \perp \vec{b}$$

Rmk

For $n \le 3$, ⑤, ⑥ are geometric properties

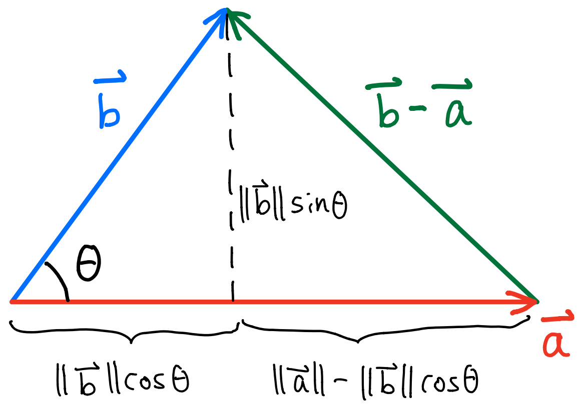
$\downarrow$ Generalize

For $n \ge 4$, ⑤, ⑥ are definitions

for $\|\vec{a}\|$ and θ

Pf of ⑥ (for $n \le 3$)

Assume $\theta \le \frac{\pi}{2}$



Note

$$\|\vec{b}-\vec{a}\|^2 = (\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a})$$
$$= \vec{b}\cdot\vec{b} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{a}$$
$$= \|\vec{b}\|^2 + \|\vec{a}\|^2 - 2\vec{a}\cdot\vec{b} \quad \text{(I)}$$

On the other hand,

$$\|\vec{b}-\vec{a}\|^2 = (\|\vec{a}\| - \|\vec{b}\|\cos\theta)^2 + (\|\vec{b}\|\sin\theta)^2$$
$$= \|\vec{a}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta + \|b\|^2\cos^2\theta$$
$$+ \|\vec{b}\|^2 \sin^2\theta$$
$$= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta \quad \text{(II)}$$

Compare (I) and (II) $\Rightarrow$ ⑥

Ex Try the case $\theta > \frac{\pi}{2}$

eg Let $\vec{v}, \vec{w}$ have same length

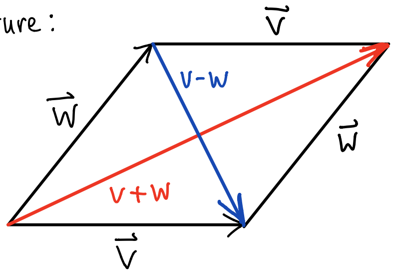
Show that $(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w})=0$

Sol

$$\begin{aligned}
\text{L.H.S.} &= (\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) \\
&= \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{v} - \vec{w}\cdot\vec{w} \\
&= \|v\|^2 - \vec{v}\cdot\vec{w} + \vec{v}\cdot\vec{w} - \|w\|^2 \\
&= \|v\|^2 - \|v\|^2 \qquad \text{(same length)} \\
&= 0
\end{aligned}$$

$\Rightarrow (\vec{v}+\vec{w}) \perp (\vec{v}-\vec{w})$

Picture:



$\|v\| = \|w\|$

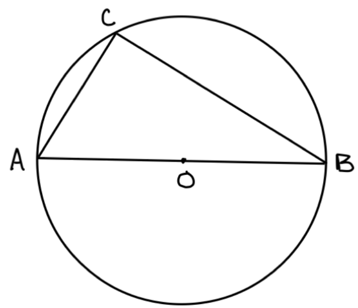
$\Rightarrow$ rhombus

> **Geometric meaning**
>
> Diagonals of a rhombus are perpendicular

eg Consider a circle centered at O.

AB is diameter.

Show that $\angle ACB = 90°$



Sol

$\vec{AC} = \vec{AO} + \vec{OC}$

$\vec{BC} = \vec{BO} + \vec{OC} = -\vec{AO} + \vec{OC}$

$$\begin{aligned}
\vec{AC}\cdot\vec{BC} &= (\vec{AO}+\vec{OC})\cdot(-\vec{AO}+\vec{OC}) \\
&= -\vec{AO}\cdot\vec{AO} + \vec{AO}\cdot\vec{OC} - \vec{OC}\cdot\vec{AO} + \vec{OC}\cdot\vec{OC} \\
&= -\|\vec{AO}\|^2 + \|\vec{OC}\|^2 \qquad \left(\begin{gathered}\|AO\| = \|OC\| \\ \text{are radius}\end{gathered}\right) \\
&= 0
\end{aligned}$$

$\therefore \vec{AC} \perp \vec{BC} \Rightarrow \angle ACB = 90°$

Two inequalities (Au 1.2)

Cauchy-Schwarz Inequality

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$. Then

$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$$

i.e. $\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum a_i^2} \cdot \sqrt{\sum b_i^2}$

Equality holds $|\vec{a} \cdot \vec{b}| = \|\vec{a}\|\|\vec{b}\|$ $\iff$ $\vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$ for some $r \in \mathbb{R}$

Rmk For $n \leq 3$,

Cauchy-Schwarz inequality follows from

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

Pf

Case 1: If $\vec{a} = \vec{0}$ or $\vec{b} = 0$

L.H.S. = R.H.S = 0

Equality holds and $\vec{a} = 0\vec{b}$ or $\vec{b} = 0\vec{a}$

Case 2: $\vec{a}, \vec{b} \neq \vec{0}$

Let $f(t) = \|t\vec{a} - \vec{b}\|^2$ for $t \in \mathbb{R}$

Then

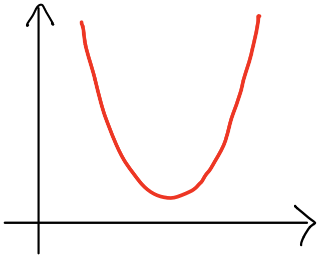
$$f(t) = (t\vec{a} - \vec{b}) \cdot (t\vec{a} - \vec{b})$$
$$= t^2 \vec{a} \cdot \vec{a} - t\vec{a} \cdot \vec{b} - t\vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$$

Note $f(t) = \|t\vec{a} - \vec{b}\|^2 \geq 0$ for any t

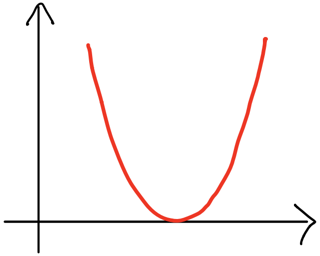
Note that

1. $f(t)$ is a real quadratic polynomial with leading coefficient $\|\vec{a}\|^2 > 0$

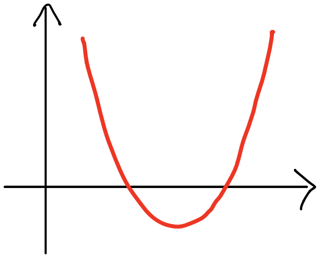
2. For a real quadratic polynomail with <u>positive</u> leading coefficient



discriminant $\Delta < 0$
(no real root)



$\Delta = 0$
(repeated root)



$\Delta > 0$
(distinct real roots)

$$f(t) = \|\vec{a}\|^2 t^2 - 2\vec{a}\cdot\vec{b}\, t + \|\vec{b}\|^2 \geq 0 \quad \forall t \in \mathbb{R}$$

$$\Rightarrow \quad \Delta \leq 0$$

$$\Rightarrow \quad (-2\vec{a}\cdot\vec{b})^2 - 4(\|\vec{a}\|^2)(\|\vec{b}\|^2) \leq 0$$

$$\Rightarrow \quad (\vec{a}\cdot\vec{b})^2 \leq \|\vec{a}\|^2 \|\vec{b}\|^2$$

$$\Rightarrow \quad |\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\,\|\vec{b}\|$$

Also,

Equality holds $\Leftrightarrow \Delta = 0$

$\Leftrightarrow$ $f(t) = \|t\vec{a} - \vec{b}\|^2$ has a repeated root

$\Leftrightarrow$ $f(r) = \|r\vec{a} - \vec{b}\|^2 = 0$ for some $r \in \mathbb{R}$

$\Leftrightarrow$ $r\vec{a} - \vec{b} = \vec{0}$ for some $r \in \mathbb{R}$

$\Leftrightarrow$ $b = r\vec{a}$ for some $r \in \mathbb{R}$

Rmk Let $\vec{a}, \vec{b} \neq \vec{0} \in \mathbb{R}^n$

We showed that if $n \leq 3$,

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\| \, \|\vec{b}\|}\right)$$

For any $n \geq 1$

Cauchy-Schwarz inequality

$$\Rightarrow -1 \leq \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\| \, \|\vec{b}\|} \leq 1$$

Hence, we can define

$$\theta = \cos^{-1}\left(\frac{a \cdot b}{\|a\| \, \|b\|}\right) \in [0, \pi]$$

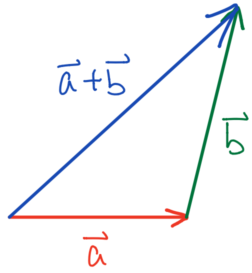
to be the angle between $\vec{a}$ and $\vec{b}$ for $n \geq 4$

Triangle Inequality

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$. Then

$$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$

Equality holds $\Leftrightarrow$ $\vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$ for some $r \geq 0$



Pf
$$\begin{aligned}
\|\vec{a} + \vec{b}\|^2 &= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) \\
&= \vec{a}\cdot\vec{a} + 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} \\
&\leq \|\vec{a}\|^2 + 2|\vec{a}\cdot\vec{b}| + \|\vec{b}\|^2 \\
&\leq \|\vec{a}\|^2 + 2\|\vec{a}\| \, \|\vec{b}\| + \|\vec{b}\|^2 \quad \text{(Cauchy-Schwarz)} \\
&= (\|\vec{a}\| + \|\vec{b}\|)^2
\end{aligned}$$

$$\Rightarrow \|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$

Equality holds $\Leftrightarrow$ $\|\vec{a}\| \, \|\vec{b}\| \overset{\text{I}}{=} |\vec{a}\cdot\vec{b}|$ $\overset{\text{II}}{=} \vec{a}\cdot\vec{b}$ $\Leftrightarrow$ $\underline{\vec{a} = r\vec{b} \text{ or } \vec{b} = r\vec{a}}$ I with $\underline{r \geq 0}$ II

Rmk Cauchy-Schwarz inequality $\Leftrightarrow$ Triangle inequality

($\Rightarrow$: Pf above, $\Leftarrow$ Ex)

Cross Product (defined only in $\mathbb{R}^3$) (Au 1.2, Thomas 12.4)

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$

Define their **cross product** to be

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

where $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$, $\hat{k} = (0,0,1)$

Recall: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

eg Let $\vec{a} = (2,3,5)$, $\vec{b} = (1,2,3)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix}$$

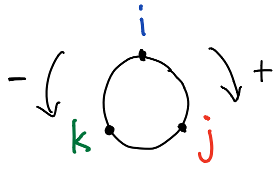
$$= \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \hat{k}$$

$$= -\hat{i} - \hat{j} + \hat{k}$$

$$= (-1, -1, 1)$$

Remark

$$\boxed{\begin{array}{lll} i \times i = \vec{0} & i \times j = k & i \times k = -j \\ j \times i = -k & j \times j = \vec{0} & j \times k = i \\ k \times i = j & k \times j = -i & k \times k = \vec{0} \end{array}}$$



Properties of Cross Product

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$

Algebraic

(follow from property of determinant)

① $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

② $(\alpha\vec{a} + \beta\vec{b}) \times \vec{c} = \alpha\vec{a}\times\vec{c} + \beta\vec{b}\times\vec{c}$

$\vec{a} \times (\beta\vec{b} + \gamma\vec{c}) = \beta\,\vec{a}\times\vec{b} + \gamma\,\vec{a}\times\vec{c}$

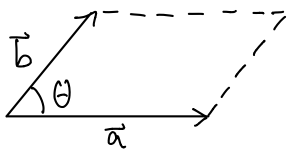
Note: ① $\Rightarrow \vec{a}\times\vec{a} = -\vec{a}\times\vec{a} \Rightarrow \vec{a}\times\vec{a} = \vec{0}$

Geometric

③ Let θ be the angle between $\vec{a}, \vec{b}$

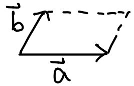
$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$

$=$ Area of parallelogram spanned by $\vec{a}, \vec{b}$



$0 \le \theta \le \pi$

$\therefore\ \vec{a}\times\vec{b} = \vec{0}$

$\Leftrightarrow$ Area () $= 0$

$\Leftrightarrow \vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$ for some $r \in \mathbb{R}$

$\Leftrightarrow \{\vec{a}, \vec{b}\}$ is linearly dependent

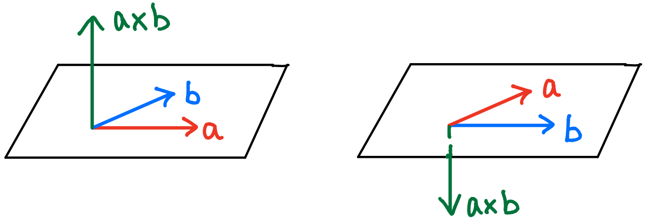
(see linear algebra)

④ $(\vec{a}\times\vec{b})\cdot\vec{a} = (\vec{a}\times\vec{b})\cdot\vec{b} = 0$ (Exercise)

If $\vec{a}\times\vec{b}$ is non-zero, then

$\Rightarrow \vec{a}\times\vec{b} \perp \vec{a}$, $\vec{a}\times\vec{b} \perp \vec{b}$

Also, $\vec{a}, \vec{b}, \vec{a}\times\vec{b}$ satisfy right hand rule



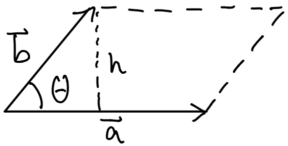
Pf of ③

By straight forward expansion

$$\|\vec{a}\times\vec{b}\|^2 = \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2(1-\cos^2\theta)$$

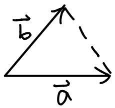
$$= \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta$$



$$\text{Area} = \|\vec{a}\|\,h$$

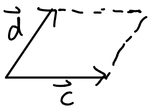
$$= \|\vec{a}\|\,\|b\|\sin\theta$$

Rmk

- Area of 

$= \frac{1}{2}\|\vec{a}\times\vec{b}\|$

- If $\vec{c},\vec{d}\in\mathbb{R}^2$, then

Area of 

$= \left|\det\begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix}\right|$

Triple Product (only in $\mathbb{R}^3$)

Let $\overbrace{\vec{a},\vec{b},\vec{c}}^{\text{order matters}}\in\mathbb{R}^3$.

Their **triple product** of $\vec{a},\vec{b},\vec{c}$ is defined to be

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\cdot(c_1\ c_2\ c_3)$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\boxed{(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}} \quad (*)$$

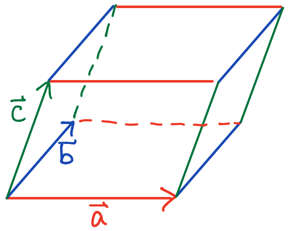
Rmk It follows from $(*)$ that

$$(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a} = (\vec{c}\times\vec{a})\cdot\vec{b}$$

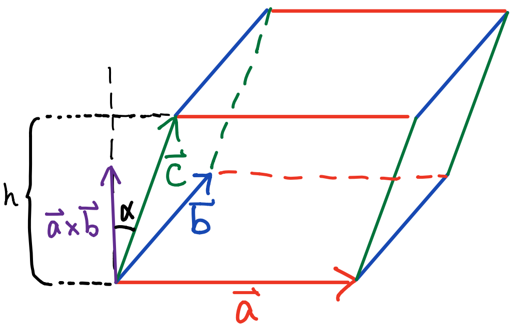
$$= -(\vec{b}\times\vec{a})\cdot c = -(\vec{c}\times\vec{b})\cdot\vec{a} = -(\vec{a}\times\vec{c})\cdot\vec{b}$$

Geometric Meaning

$$\left|(\vec{a}\times\vec{b})\cdot\vec{c}\right| = \text{Volume of parallelepiped spanned by } \vec{a},\vec{b},\vec{c}$$



Pf

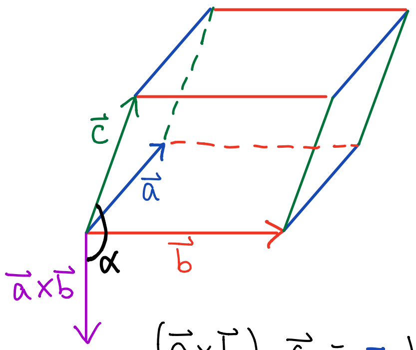


$\left(\alpha \leq \frac{\pi}{2}\right)$

For $\alpha \leq \frac{\pi}{2}$,

$$\begin{aligned}(\vec{a}\times\vec{b})\cdot\vec{c} &= \|\vec{a}\times\vec{b}\|\,\|\vec{c}\|\cos\alpha \\ &= \|\vec{a}\times\vec{b}\|\,h \\ &= \text{Base Area} \times \text{height} \\ &= \text{Volume of parallelepiped}\end{aligned}$$

The case for $\frac{\pi}{2} < \alpha \leq \pi$ is similar



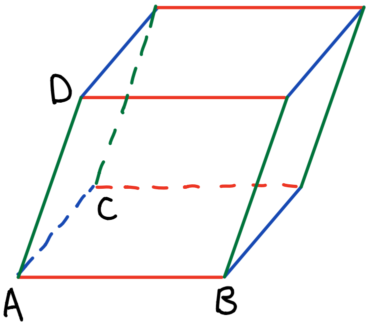
$$(\vec{a}\times\vec{b})\cdot\vec{c} = -\text{Volume of parallelepiped}$$

Rmk

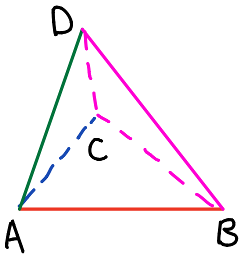
$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \iff \text{Volume of parallelepiped} = 0$$

$$\iff \{\vec{a},\vec{b},\vec{c}\} \text{ is lin dependent}$$

Given 4 points A, B, C, D in $\mathbb{R}^3$



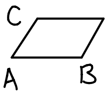
Parallelepiped



Tetrahedron

Volume of Tetrahedron

$= \frac{1}{3}$ Area $(\triangle ABC) \cdot$ height

$= \frac{1}{3} \cdot \frac{1}{2}$ Area () $\cdot$ height

$= \frac{1}{6}$ Volume of Parallelepiped

eg Let $A = (1,0,1)$ $B = (1,1,2)$

$C = (2,1,1)$ $D = (2,1,3)$

Find volume of tetrahedron $ABCD$

Sol

$\vec{AB} = (1,1,2) - (1,0,1) = (0,1,1)$

$\vec{AC} = (1,1,0)$

$\vec{AD} = (1,1,2)$

$$(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = -2$$

$\Rightarrow$ volume of tetrahedron $= \frac{1}{6}|-2| = \frac{1}{3}$.